

PERTURBED EQUATIONS OF MOTION FOR FORMATION FLIGHT NEAR THE SUN-EARTH L_2 POINT

A. M. Segerman *
M. F. Zedd †

Abstract

NASA is planning missions to the vicinity of the Sun-Earth L_2 point, some involving a distributed system of telescope spacecraft, configured in a plane about a hub. Several sets of differential equations are written for the formation flight of such telescopes relative to the hub, with varying levels of fidelity. Effects are cast as additive perturbations to the circular restricted three-body problem, expanded in terms of the system distances, to an accuracy of 10–20 m. These include Earth's orbital eccentricity, lunar motion, solar radiation pressure, and small thrusting forces. Simulations validating the expanded differential equations are presented.

INTRODUCTION

NASA's Goddard Space Flight Center is planning a series of missions in the vicinity of the Sun-Earth L_2 libration point. Some of these projects will involve a distributed space system of telescope spacecraft acting together as a single telescope for high-resolution. The individual telescopes will be configured in a plane, surrounding a hub, where the telescope plane can be aimed toward various astronomical targets of interest. Nominal missions locate the hub in a halo orbit, at distances of hundreds of thousands of kilometers from the L_2 point. The individual telescopes are to be placed at less than one kilometer from the hub.

The present research continues the earlier work of Segerman and Zedd [1]. In that paper, the circular restricted three-body problem was used as a basis for the derivation of differential equations which describe the motion of a typical telescope spacecraft relative to a nearby hub. Following the approach of Richardson [2], a modified Lindstedt-Poincaré method was used in order to develop an uncontrolled periodic solution for this relative motion, which is a requirement for maintaining the telescope formation. The solution

*Astrodynamics Engineer, AT&T Government Solutions, Inc., at Naval Research Laboratory, Code 8233, 4555 Overlook Ave. SW, Washington, DC 20375-5355, (202) 767-3819, fax: (202) 404-7785, email: segerman@nrl.navy.mil

†Acrospace Engineer, Mathematics and Orbit Dynamics Section, Naval Research Laboratory, Code 8233, 4555 Overlook Ave. SW, Washington, DC 20375-5355, (202) 404-8337, fax: (202) 404-7785, email: zedd@nrl.navy.mil

included linear effects of the hub motion about L_2 ; quadratic hub motion effects were used to develop relationships between the telescope frequency and amplitude. In the course of that analysis, a halo-type orbit of the telescope about the hub was chosen to provide periodic motion in the aperture plane. Comparison was made between the resulting relative motion solution and a baseline numerical solution of the full circular restricted equations of motion.

This research furthers the earlier work by adding the increased fidelity associated with perturbations to the circular restricted three-body problem. Each of the perturbing effects is cast as additive perturbations to the circular restricted equations of motion. For the purpose of this analysis, these perturbations include the ellipticity of Earth's orbit about Sun, the orbiting Moon, and solar radiation pressure.

In considering the effects of the elliptical restricted problem, it is noted that the various relative motion vectors within the three-body system are ordinarily functionally dependent upon Earth's orbital eccentricity. To avoid this dependence, the vectors are defined relative to reference positions of Sun, Earth, and L_2 along the line of syzygy. Then, the relative equations of motion for an individual telescope are expanded in terms of the distance of the hub from L_2 as well as the distance of the telescope from the hub. These expansions are written so as to include all terms which contribute at the 10-20 meter level or above. Explicit linear contributions of Earth's orbital eccentricity are retained; higher order contributions are excluded, as being below the prescribed level of fidelity.

For the inclusion of the lunar contribution, expansions of the vector forces are formed, here involving the small-scale distance of a telescope from the hub, as compared to the larger scale of distances of the lunar motion within the Sun-Earth system. Again, the expansions are truncated in a fashion consistent with the previously stated fidelity level.

Solar radiation contributions are included as well. These terms include the effects of possible eclipses resulting from various geometric configurations of the system. This effect is included both for completeness and with the awareness that it may be possible to actively adjust the spacecraft attitude in order to use the solar radiation pressure as part of an orbital control scheme.

Finally, the equations are written to include the presence of small body-fixed thrusting forces, which are assumed to be used as part of the orbit control.

Simulations were conducted, comparing numerical integration of the various resulting sets of differential equations with numerical integration of the full, unexpanded differential equations. These simulations have been used to verify the validity of the expanded equations. Some of the results of these comparisons are presented. Studies of the solution's sensitivity to errors in hub position knowledge are also discussed.

EQUATIONS OF MOTION

Circular Restricted Three-Body Problem

In the preliminary phase of this research [1], the general second order differential equations of motion were constructed for an object near the Sun-Earth L_2 libration point, using the force model of the classical circular restricted three-body problem. In this model, Earth is treated as being in a circular orbit about the sun, the spacecraft mass is considered to be negligible as compared to the two primaries, and only point-mass gravitational forces are considered.

For this system, depicted in Figure 1, the differential equations of motion for an object

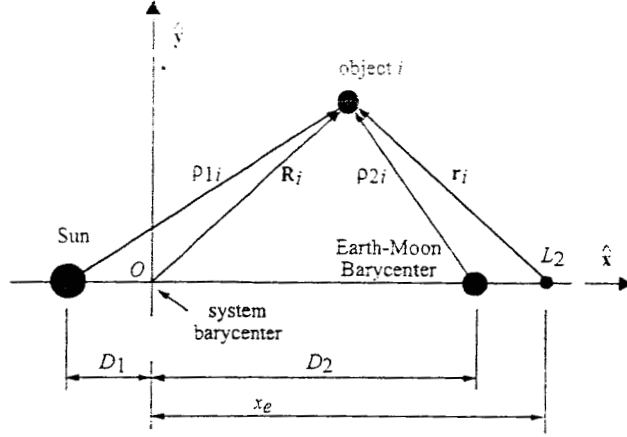


Figure 1 Coordinate Axis Definition

(object i) near the Sun-Earth L_2 are given by

$$\ddot{\mathbf{r}}_i = - \left(\frac{\mu_1}{\rho_{1i}^3} + \frac{\mu_2}{\rho_{2i}^3} \right) \mathbf{r}_i - \left(\frac{\mu_1(x_e + D_1)}{\rho_{1i}^3} + \frac{\mu_2(x_e - D_2)}{\rho_{2i}^3} \right) \hat{\mathbf{x}} + n^2 x_e \hat{\mathbf{x}},$$

where

\mathbf{r}_i = vector from L_2 to object i

μ_1 = solar Keplerian constant

μ_2 = terrestrial Keplerian constant (Earth + Moon)

ρ_{1i} = distance from Sun to object i

ρ_{2i} = distance from Earth-Moon barycenter to object i

x_e = distance from system barycenter to L_2

D_1 = distance from system barycenter to Sun

D_2 = distance from system barycenter to Earth-Moon barycenter

$\hat{\mathbf{x}}$ = unit vector parallel to Sun-Earth line of syzygy,

pointing in Sun-to-Earth direction

n = terrestrial mean motion about Sun (assumed constant).

The coordinate frame of Figure 1 is a rotating reference frame with origin O at the system barycenter. The x -axis points along the Sun-Earth line of syzygy and the z -axis is parallel to the system angular momentum; the y -axis completes the dextral coordinate system.

Let \mathbf{r}_h and \mathbf{r}_t denote, respectively, the vector from L_2 to the hub and to a telescope. Therefore, if \mathbf{r} is the vector from the hub to the telescope, the differential equation of motion for the telescope relative to the hub is

$$\begin{aligned} \ddot{\mathbf{r}} = & -\mu_1 \left(\frac{\mathbf{r}_t}{\rho_{1t}^3} - \frac{\mathbf{r}_h}{\rho_{1h}^3} \right) - \mu_2 \left(\frac{\mathbf{r}_t}{\rho_{2t}^3} - \frac{\mathbf{r}_h}{\rho_{2h}^3} \right) \\ & - \mu_1(x_e + D_1) \left(\frac{1}{\rho_{1t}^3} - \frac{1}{\rho_{1h}^3} \right) \hat{\mathbf{x}} - \mu_2(x_e - D_2) \left(\frac{1}{\rho_{2t}^3} - \frac{1}{\rho_{2h}^3} \right) \hat{\mathbf{x}}, \end{aligned} \quad (1)$$

where now, in general, the subscripts h and t refer to the hub and telescope.

Elliptical Restricted Three-Body Problem

The extension to Equation (1) is now developed for the case of the elliptical restricted three-body problem. First, it is necessary to locate the point which is analogous to the libration point L_2 in the circular restricted problem. As would be expected, such a point exists; as the Sun-Earth distance varies, the location of the point oscillates along the x -axis.

Elliptical Problem Libration Point Analog. Say that there is an elliptical analog to the L_2 point and that its position relative to the (assumed inertial) system barycenter is given by

$$\mathbf{R}_e = x_e \hat{\mathbf{x}} + y_e \hat{\mathbf{y}} + z_e \hat{\mathbf{z}}.$$

Letting f refer to the true anomaly of Earth's orbit about the sun, the coordinate system has angular velocity

$$\omega_c = f \hat{\mathbf{z}},$$

which now is considered to be varying throughout the year. Differentiating \mathbf{R}_e ,

$$\dot{\mathbf{R}}_e = \begin{bmatrix} \dot{x}_e - f y_e \\ \dot{y}_e + f x_e \\ \dot{z}_e \end{bmatrix} \quad \text{and} \quad \ddot{\mathbf{R}}_e = \begin{bmatrix} \ddot{x}_e - \ddot{f} y_e - 2f \dot{y}_e - f^2 x_e \\ \ddot{y}_e + \ddot{f} x_e + 2f \dot{x}_e - f^2 y_e \\ \ddot{z}_e \end{bmatrix},$$

where the column vector notation is used to indicate the xyz vector components.

The Newtonian gravitational force per unit mass acting on an object at this point by Sun and Earth is given by

$$\mathbf{F}_e/m = \begin{bmatrix} -\frac{\mu_1(x_e + D_1)}{r_{1e}^3} - \frac{\mu_2(x_e - D_2)}{r_{2e}^3} \\ -\frac{\mu_1 y_e}{r_{1e}^3} - \frac{\mu_2 y_e}{r_{2e}^3} \\ -\frac{\mu_1 z_e}{r_{1e}^3} - \frac{\mu_2 z_e}{r_{2e}^3} \end{bmatrix},$$

where

$$\mathbf{r}_{1e} = (x_e + D_1) \hat{\mathbf{x}} + y_e \hat{\mathbf{y}} + z_e \hat{\mathbf{z}} \quad \text{and} \quad \mathbf{r}_{2e} = (x_e - D_2) \hat{\mathbf{x}} + y_e \hat{\mathbf{y}} + z_e \hat{\mathbf{z}},$$

and D_1 and D_2 now refer to the time-varying locations of Sun and Earth along the line of syzygy.

Clearly, as in the circular restricted case, the z equation is decoupled, and admits a solution $z_e = 0$. If, as anticipated, the desired solution involves $y_e = 0$, the x and y components of the force-acceleration equation become:

$$\begin{aligned} \hat{\mathbf{x}} : \ddot{x}_e - f^2 x_e + \frac{\mu_1}{(x_e + D_1)^2} + \frac{\mu_2}{(x_e - D_2)^2} &= 0 \\ \hat{\mathbf{y}} : \ddot{y}_e + 2f \dot{x}_e &= \frac{1}{x_e} \frac{d}{dt} (x_e^2 \dot{f}) = 0. \end{aligned}$$

From the y -component equation, $x_e^2 \dot{f}$ is therefore constant. However, conservation of the Sun-Earth two-body angular momentum h per unit mass gives

$$D^2 \dot{f} = h,$$

where D is the varying Sun-Earth distance $D_1 + D_2$. Of course, unlike the case of the circular restricted problem, this distance varies throughout the year. Substitution for \dot{f} in the y -component equation gives

$$\left(\frac{x_e}{D}\right)^2 = \text{constant}.$$

For later convenience, define this constant in terms of the constant γ , such that

$$\left(\frac{x_e}{D}\right)^2 = \left(\gamma + \frac{D_2}{D}\right)^2.$$

Taking the positive root, this gives the definition of γ as

$$\gamma \triangleq \frac{x_e - D_2}{D}.$$

In the case of the Sun-Earth system, γ is approximately 0.01007824; x_e varies between 1.486×10^8 and 1.536×10^8 km throughout the course of the year.

Relative Motion Equation Derivation. Let \mathbf{R}_i denote the position vector from the system barycenter (point O) to spacecraft i . Referring again to Figure 1,

$$\mathbf{R}_i = x_e \hat{\mathbf{x}} + \mathbf{r}_i.$$

Differentiating,

$$\begin{aligned}\dot{\mathbf{R}}_i &= \dot{x}_e \hat{\mathbf{x}} + \dot{f} x_e \hat{\mathbf{y}} + \dot{\mathbf{r}}_i \\ \ddot{\mathbf{R}}_i &= (\ddot{x}_e - \dot{f}^2 x_e) \hat{\mathbf{x}} + (2\dot{f} \dot{x}_e + \ddot{f} x_e) \hat{\mathbf{y}} + \ddot{\mathbf{r}}_i.\end{aligned}$$

The distance x_e and its time derivatives may be written in terms of the varying Sun-Earth distance D and its derivatives, using the following definitions of the constants γ and ρ , and the associated relationships:

$$\begin{aligned}\gamma &\triangleq \frac{x_e - D_2}{D} & \rho &\triangleq \frac{\mu_2}{\mu} & D_1 &= \rho D \\ \gamma + 1 &= \frac{x_e + D_1}{D} & 1 - \rho &= \frac{\mu_1}{\mu} & D_2 &= (1 - \rho) D,\end{aligned}$$

where

$$\mu = \mu_1 + \mu_2.$$

Then,

$$\begin{aligned}x_e &= \gamma D + D_2 \\ &= (\gamma + 1 - \rho) D, & \dot{x}_e &= (\gamma + 1 - \rho) \dot{D}, & \ddot{x}_e &= (\gamma + 1 - \rho) \ddot{D}.\end{aligned}$$

This gives the acceleration vector $\ddot{\mathbf{R}}_i$ as

$$\ddot{\mathbf{R}}_i = (\gamma + 1 - \rho)(\ddot{D} - \dot{f}^2 D)\hat{\mathbf{x}} + (\gamma + 1 - \rho)(2\dot{f}\dot{D} + \ddot{f}D)\hat{\mathbf{y}} + \ddot{\mathbf{r}}_i. \quad (2)$$

Similarly, two-body relationships may be used to write \dot{D} and \ddot{D} in terms of \dot{f} and \ddot{f} . Using the two-body equations of motion for the sun (could use Earth):

$$\begin{aligned} \mathbf{R}_1 &= -D_1\hat{\mathbf{x}} = -\rho D\hat{\mathbf{x}} \\ \dot{\mathbf{R}}_1 &= -\rho\dot{D}\hat{\mathbf{x}} - \rho\dot{f}D\hat{\mathbf{y}} \\ \ddot{\mathbf{R}}_1 &= \rho(\dot{f}^2 D - \ddot{D})\hat{\mathbf{x}} - \rho(\ddot{f}D + 2\dot{f}\dot{D})\hat{\mathbf{y}} \\ &= \frac{\mu_2}{D^2}\hat{\mathbf{x}} \quad (\text{two-body force}) \end{aligned}$$

This gives the component equations

$$\begin{aligned} \hat{\mathbf{x}} : \rho(\dot{f}^2 D - \ddot{D}) &= \frac{\mu_2}{D^2} \\ \hat{\mathbf{y}} : -\rho(\ddot{f}D + 2\dot{f}\dot{D}) &= -\frac{\rho}{D} \frac{d}{dt}(D^2 \dot{f}) = 0, \end{aligned}$$

which yield

$$\ddot{D} - \dot{f}^2 D = -\frac{\mu_2}{\rho D^2} \quad \text{and} \quad 2\dot{f}\dot{D} + \ddot{f}D = 0.$$

Substituting into Equation (2), the acceleration becomes

$$\ddot{\mathbf{R}}_i = \left[-(\gamma + 1) \frac{\mu_1}{D^2} - \gamma \frac{\mu_2}{D^2} \right] \hat{\mathbf{x}} + \ddot{\mathbf{r}}_i.$$

Introducing the two-body forces,

$$\ddot{\mathbf{R}}_i = \left[-(\gamma + 1) \frac{\mu_1}{D^2} - \gamma \frac{\mu_2}{D^2} \right] \hat{\mathbf{x}} + \ddot{\mathbf{r}}_i = -\frac{\mu_1}{\rho_{1i}^3} \rho_{1i} - \frac{\mu_2}{\rho_{2i}^3} \rho_{2i}. \quad (3)$$

Note that the explicit appearance of the time derivatives of f has now been removed.

Next, the vectors ρ_{1i} and ρ_{2i} may be written as

$$\begin{aligned} \rho_{1i} &= (x_e + D_1)\hat{\mathbf{x}} + \mathbf{r}_i \\ &= (\gamma + 1)D\hat{\mathbf{x}} + \mathbf{r}_i \end{aligned} \quad \text{and} \quad \begin{aligned} \rho_{2i} &= (x_e - D_2)\hat{\mathbf{x}} + \mathbf{r}_i \\ &= \gamma D\hat{\mathbf{x}} + \mathbf{r}_i. \end{aligned}$$

Substituting into Equation (3),

$$\ddot{\mathbf{R}}_i - \left(\frac{\mu_1}{\rho_{1i}^3} + \frac{\mu_2}{\rho_{2i}^3} \right) \mathbf{r}_i = \left[(\gamma + 1) \frac{\mu_1}{\rho_{1i}^3} + \gamma \frac{\mu_2}{\rho_{2i}^3} \right] D\hat{\mathbf{x}}.$$

In the rotating frame, \mathbf{r}_i and its time derivatives are given by

$$\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \quad \dot{\mathbf{r}}_i = \begin{bmatrix} \dot{x}_i - \dot{f}y_i \\ \dot{y}_i + \dot{f}x_i \\ \dot{z}_i \end{bmatrix}, \quad \ddot{\mathbf{r}}_i = \begin{bmatrix} \ddot{x}_i - \ddot{f}y_i - 2\dot{f}\dot{y}_i - \dot{f}^2 x_i \\ \ddot{y}_i + \ddot{f}x_i + 2\dot{f}\dot{x}_i - \dot{f}^2 y_i \\ \ddot{z}_i \end{bmatrix}.$$

Note that vectors ρ_{1i} , ρ_{2i} , and \mathbf{r}_i are functions of the eccentricity of Earth's orbit, as is D . To avoid this dependence, redefine these vectors relative to reference positions of Sun, Earth, and L_2 along the line of syzygy. Denoting the reference distances by an overbar,

$$\begin{aligned}\mathbf{R}_i &= \bar{x}_e \hat{\mathbf{x}} + \mathbf{r}_i \\ \dot{\mathbf{R}}_i &= f \bar{x}_e \hat{\mathbf{y}} + \dot{\mathbf{r}}_i \\ \ddot{\mathbf{R}}_i &= -f^2 \bar{x}_e \hat{\mathbf{x}} + \ddot{f} \bar{x}_e \hat{\mathbf{y}} + \ddot{\mathbf{r}}_i.\end{aligned}$$

Again using the definitions of γ and ρ , $\bar{x}_e = (\gamma + 1 - \rho) \bar{D}$.

Using this form of the acceleration,

$$\begin{aligned}\ddot{\mathbf{R}}_i &= (\gamma + 1 - \rho)(-f^2 \hat{\mathbf{x}} + \ddot{f} \hat{\mathbf{y}}) \bar{D} + \ddot{\mathbf{r}}_i \\ &= -\left(\frac{\mu_1}{\rho_{1i}^3} + \frac{\mu_2}{\rho_{2i}^3}\right) \mathbf{r}_i - \left[(\gamma + 1) \frac{\mu_1}{\rho_{1i}^3} + \gamma \frac{\mu_2}{\rho_{2i}^3}\right] \bar{D} \hat{\mathbf{x}},\end{aligned}\quad (4)$$

where

$$\rho_{1i} = (\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_i \quad \text{and} \quad \rho_{2i} = \gamma \bar{D} \hat{\mathbf{x}} + \mathbf{r}_i.$$

Vectors \mathbf{r}_i and $\dot{\mathbf{r}}_i$ take the same forms as before, now relative to the reference L_2 location.

For the relative motion, let

$$\mathbf{r} = \mathbf{r}_t - \mathbf{r}_h = \mathbf{R}_t - \mathbf{R}_h.$$

Then,

$$\begin{aligned}\ddot{\mathbf{r}} &= -\mu_1 \left(\frac{\mathbf{r}_t}{\rho_{1t}^3} - \frac{\mathbf{r}_h}{\rho_{1h}^3} \right) - \mu_2 \left(\frac{\mathbf{r}_t}{\rho_{2t}^3} - \frac{\mathbf{r}_h}{\rho_{2h}^3} \right) \\ &\quad - \left[\mu_1 (\gamma + 1) \left(\frac{1}{\rho_{1t}^3} - \frac{1}{\rho_{1h}^3} \right) + \mu_2 \gamma \left(\frac{1}{\rho_{2t}^3} - \frac{1}{\rho_{2h}^3} \right) \right] \bar{D} \hat{\mathbf{x}}.\end{aligned}\quad (5)$$

Note the similar form to Equation (1).

Lunar Gravitational Effects

This section discusses the lunar contribution to the telescope motion relative to the hub. Terms corresponding to the gravitational force of the moon upon the hub and telescope spacecraft are included in the relative equations of motion (telescope relative to hub). The resulting contribution is then cast as an additive perturbation to the elliptical restricted three-body problem equations, such that when the lunar motion is ignored, the contribution reverts to the lunar mass placed at the Earth position. The moon is treated as a point mass.

As shown in Figure 2, let ρ_{3i} denote the vector from the moon to spacecraft i , either the hub or a telescope, and let μ_3 denote the lunar Keplerian constant. The lunar force per unit mass on spacecraft i is then given by

$$-\mu_3 \frac{\rho_{3i}}{\rho_{3i}^3}.\quad (6)$$

Therefore, the contribution to the equations of motion of the telescope relative to the hub is

$$\mathbf{F}_m = -\mu_3 \left(\frac{\rho_{3t}}{\rho_{3t}^3} - \frac{\rho_{3h}}{\rho_{3h}^3} \right),\quad (7)$$

where the subscripts h and t refer respectively to the hub and telescope.

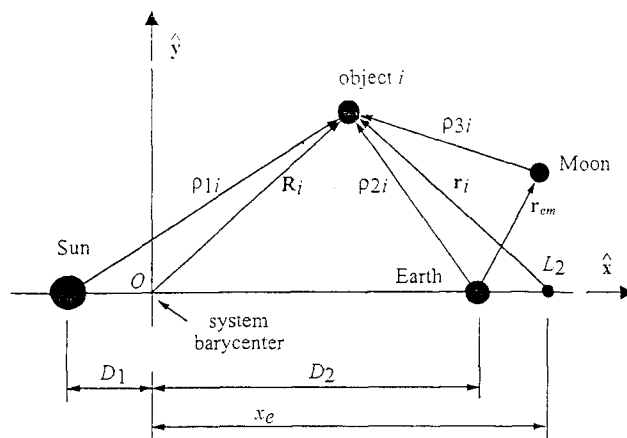


Figure 2 Coordinate Axis Definition (including Moon)

Solar Radiation Pressure Effects

The pressure from solar radiation imparts a relatively tiny force on a telescope spacecraft. Depending upon spacecraft design and distance from the sun, the force can perturb both the spacecraft's attitude and orbit. The force can also be harnessed to beneficially propel the spacecraft. The model presented here provides the force on a spacecraft, accounting for the force reduction when the spacecraft orbits through the terrestrial shadow.

At a distance ρ_{1t} from the sun, the solar flux I (the irradiance) acting on the spacecraft is given by

$$I = \frac{L}{4\pi\rho_{1t}^2},$$

where

$$L = 3.842 \times 10^{26} \text{ watts}$$

is the solar luminosity (total emitted radiation).

If A is the cross-sectional area of a spacecraft of mass m projected normal to the spacecraft-Sun line, then the solar radiation force F_s per unit mass acting on the spacecraft is

$$F_s = \frac{C_R L A}{4\pi m c \rho_{1t}^2} = \frac{1.0198 \times 10^{17} C_R A}{m \rho_{1t}^2} \text{ N/unit mass,}$$

where c is the speed of light and $0 \leq C_R \leq 2$ is the parameter characteristic of the reflectivity of the spacecraft surface facing the sun:

$$C_R = 0 \quad \text{translucent,}$$

$$C_R = 1 \quad \text{perfectly absorbent,}$$

$$C_R = 2 \quad \text{perfectly reflective.}$$

For a trajectory which passes through any portion of Earth's shadow, the full disk of the sun will be partially obscured. In the vicinity of L_2 , this will occur at distances normal to the line of syzygy of approximately 13,420 km or less. In such cases, the force expression above must be scaled by a "luminosity reduction factor" σ which ranges from zero (total

eclipse) to unity (full sunlight). The appropriate expression for the force per unit mass is then written:

$$\mathbf{F}_s = \frac{1.0198 \times 10^{17} C_R A \sigma}{m \rho_{1t}^3} \rho_{1t} = \frac{1.0198 \times 10^{17} C_R A \sigma}{m \|(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_t\|^3} [(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_t]. \quad (8)$$

Spacecraft Thruster Effects

This section presents the incorporation of the effects of body-mounted thrusters in the equations of motion.

First, the body-fixed acceleration components imparted by the thrusters are given as

$$\ddot{x}_b = F_x/m, \quad \ddot{y}_b = F_y/m, \quad \ddot{z}_b = F_z/m,$$

where F_x, F_y, F_z are the components of the thrust $\mathbf{F}_{\text{thrust}}$ in the body-fixed frame; m is the vehicle mass.

Consider an arbitrary alignment of a body-fixed coordinate frame with respect to the rotating x, y, z frame. This body-fixed x_b, y_b, z_b coordinate frame has its origin at the spacecraft's mass center.

The components of thrust expressed in the body frame must be transformed into the rotating reference frame in order to correctly incorporate these forces into the description of the motion. This transformation is expressed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix},$$

where T is the transformation matrix formed as a combination of Euler rotations.

It is somewhat easier to visualize the individual rotations by considering the inverse rotation description, rotating instead from the rotating reference frame to that fixed in the spacecraft. By first assuming the coincident alignment of both the body frame and the rotating reference frame, the inverse transformation matrix T^{-1} is then a combination of Euler rotations through a set of Euler angles:

1. first, rotate angle $(\psi + f)$ about the z_b axis, where $f = \dot{f}(t - t_0)$
2. second, rotate angle θ about the new orientation of the y_b axis
3. third, rotate angle ϕ about new orientation of the x_b axis

Combine the sequence of rotations as follows with a right-to-left ordering of the rotation matrices as

$$\begin{aligned} T^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & s(\phi) \\ 0 & -s(\phi) & c(\phi) \end{bmatrix} \begin{bmatrix} c(\theta) & 0 & -s(\theta) \\ 0 & 1 & 0 \\ s(\theta) & 0 & c(\theta) \end{bmatrix} \begin{bmatrix} c(\psi + f) & s(\psi + f) & 0 \\ -s(\psi + f) & c(\psi + f) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= T_x^{-1}(\phi) T_y^{-1}(\theta) T_z^{-1}(\psi + f), \end{aligned}$$

where the functions c and s represent cosine and sine, respectively; the rotation matrix T_i^{-1} refers to the required Euler rotation matrix about the i_b -axis. Finally, the desired

transformation matrix T (from the body-fixed frame to the rotating frame) is expressed as the inverse of T^{-1} :

$$T = (T^{-1})^{-1} = T_z(\psi + f)T_y(\theta)T_x(\phi).$$

These expressions premultiply the thrust force per unit mass acting upon the telescope spacecraft.

Note that typical accelerations due to thrust and solar radiation pressure are several orders of magnitude smaller than those due to terrestrial and solar gravity. It is noteworthy that solar radiation pressure can be used for orbit control because the force from solar radiation pressure may be comparable to that from a thruster suite.

Summary

The force equations derived above are now combined. The resulting equation models the elliptical restricted three-body problem, incorporating lunar, solar radiation, and thrust perturbations.

Combining the expressions of this section, given by Equations (4), (6), and (8), along with the thrust expressions above, the differential equations of motion of the telescope are given by:

$$\begin{aligned} \ddot{\mathbf{R}}_t &= (\gamma + 1 - \rho)(-f^2\hat{\mathbf{x}} + \ddot{f}\hat{\mathbf{y}})\bar{D} + \ddot{\mathbf{r}}_t \\ &= -\left(\frac{\mu_1}{\rho_{1t}^3} + \frac{\mu_2}{\rho_{2t}^3}\right)\mathbf{r}_t - \left[(\gamma + 1)\frac{\mu_1}{\rho_{1t}^3} + \gamma\frac{\mu_2}{\rho_{2t}^3}\right]\bar{D}\hat{\mathbf{x}} \quad \left. \vphantom{\ddot{\mathbf{R}}_t} \right\} \text{ER3B} \\ &\quad - \mu_3 \frac{\rho_{3t}}{\rho_{3t}^3} \quad \left. \vphantom{\ddot{\mathbf{R}}_t} \right\} \text{lunar} \\ &\quad + \frac{1.0198 \times 10^{17} C_{RA}\sigma}{m \|(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_t\|^3} [(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_t] \quad \left. \vphantom{\ddot{\mathbf{R}}_t} \right\} \text{SRP} \\ &\quad + \begin{bmatrix} c(\psi') & -s(\psi') & 0 \\ s(\psi') & c(\psi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix} \frac{\mathbf{F}_{\text{thrust}}}{m}, \quad \left. \vphantom{\ddot{\mathbf{R}}_t} \right\} \text{thrust} \end{aligned}$$

where $\psi' = \psi + f$. The corresponding equations for the hub are

$$\begin{aligned} \ddot{\mathbf{R}}_h &= (\gamma + 1 - \rho)(-f^2\hat{\mathbf{x}} + \ddot{f}\hat{\mathbf{y}})\bar{D} + \ddot{\mathbf{r}}_h \\ &= -\left(\frac{\mu_1}{\rho_{1h}^3} + \frac{\mu_2}{\rho_{2h}^3}\right)\mathbf{r}_h - \left[(\gamma + 1)\frac{\mu_1}{\rho_{1h}^3} + \gamma\frac{\mu_2}{\rho_{2h}^3}\right]\bar{D}\hat{\mathbf{x}} \quad \left. \vphantom{\ddot{\mathbf{R}}_h} \right\} \text{ER3B} \\ &\quad - \mu_3 \frac{\rho_{3h}}{\rho_{3h}^3} \quad \left. \vphantom{\ddot{\mathbf{R}}_h} \right\} \text{lunar} \\ &\quad + \frac{1.0198 \times 10^{17} C_{RA}\sigma}{m \|(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h\|^3} [(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h]. \quad \left. \vphantom{\ddot{\mathbf{R}}_h} \right\} \text{SRP} \end{aligned}$$

Combining the relative motion expressions of this section (Equations (5) and (7)), along with the solar radiation pressure of Equation (8) and the earlier thrust expressions, the

differential equations of relative motion for a telescope spacecraft are then given by:

$$\begin{aligned}
\ddot{\mathbf{r}} = & -\mu_1 \left(\frac{\mathbf{r}_t}{\rho_{1t}^3} - \frac{\mathbf{r}_h}{\rho_{1h}^3} \right) - \mu_2 \left(\frac{\mathbf{r}_t}{\rho_{2t}^3} - \frac{\mathbf{r}_h}{\rho_{2h}^3} \right) & \left. \begin{aligned} & \\ & - \left[\mu_1(\gamma + 1) \left(\frac{1}{\rho_{1t}^3} - \frac{1}{\rho_{1h}^3} \right) + \mu_2 \gamma \left(\frac{1}{\rho_{2t}^3} - \frac{1}{\rho_{2h}^3} \right) \right] \bar{D} \hat{\mathbf{x}} \end{aligned} \right\} \text{ER3B} \\
& - \mu_3 \left(\frac{\rho_{3t}}{\rho_{3t}^3} - \frac{\rho_{3h}}{\rho_{3h}^3} \right) & \left. \begin{aligned} & \\ & \end{aligned} \right\} \text{lunar} \\
& + \frac{1.0198 \times 10^{17} C_R A \sigma}{m \|(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_t\|^3} [(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_t] & \left. \begin{aligned} & \\ & \end{aligned} \right\} \text{SRP}_t \\
& - \frac{1.0198 \times 10^{17} C_R A \sigma}{m \|(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_h\|^3} [(\gamma + 1) \bar{D} \hat{\mathbf{x}} + \mathbf{r}_h] & \left. \begin{aligned} & \\ & \end{aligned} \right\} \text{SRP}_h \\
& + \begin{bmatrix} c(\psi') & -s(\psi') & 0 \\ s(\psi') & c(\psi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix} \frac{\mathbf{F}_{\text{thrust}}}{m} & \left. \begin{aligned} & \\ & \end{aligned} \right\} \text{thrust}
\end{aligned}$$

In the MAXIM or similar missions, there may or may not be an actual hub spacecraft located at the aperture's center. Regardless, it is necessary to treat the hub as a central reference point for locating the positions of the individual telescope spacecraft. However, unlike the gravitational forces present, the solar radiation pressure effect upon a spacecraft with actual mass and area is substantial as compared to that upon a "phantom" hub. Therefore, for purposes of simulating the solar radiation pressure, the hub "spacecraft" is treated as having the same physical characteristics as the telescope. In this manner, the hub is maintained as an adequate reference for the position of the telescope.

An example is presented in order to permit comparison of the relative contribution of the terms to the telescope motion. The initial conditions are listed in Table 1; they were taken from the examples discussed in the earlier phase of the research. As mentioned in that paper, these values were selected so as to excite only the oscillatory linear modes. The relevant physical constants are presented in Table 4 below; the computed value of \bar{D} may be found in Table 5.

Figure 3 depicts the solution to numerical integration of four different force models selected from the summary equation above. The models were applied to both the hub and telescope, and the resulting state vectors were differenced in order to determine the position of the telescope relative to the hub. In each case, the same force model was applied to both the hub and telescope. Additionally, the same value of the reference distance \bar{D} was used for all force models.

The solutions are displayed relative to the circular restricted model solution, which is obtained by including the elliptical restricted model (ER3B) terms, but with Earth's orbital eccentricity treated as being zero; this duplicates the model discussed in the earlier research. The other models depicted in Figure 3 represent the addition of ellipticity (ER3B), lunar, and solar radiation (SRP) effects over 20 days. It is noted that the elliptical contribution provides the dominant perturbation to the circular restricted solution. In this example, should the viewing of a scientific target be limited to no more than 10 days, the perturbation due to the elliptical contribution is less than 1 m.

The effects of thrusters are not simulated here, due to the vast uncertainties of force (both magnitude and direction) and duration.

Table 1
Example Initial Conditions

	hub (state rel. to L_2)	telescope (state rel. to L_2)	telescope (state rel. to hub)
$x(0)$ (km)	-227,219.419	-227,219.483	-0.064780
$y(0)$ (km)	0.0	0.0	0.0
$z(0)$ (km)	-250,000.000	-249,999.974	0.026445
$\dot{x}(0)$ (km/day)	0.0	0.0	0.0
$\dot{y}(0)$ (km/day)	25,625.039	25,625.044	0.004421
$\dot{z}(0)$ (km/day)	0.0	0.0	0.0
mass m (kg)	500	500	
F_{thrust} (N)	0	0	
sun-facing area A (m ²)	150	150	

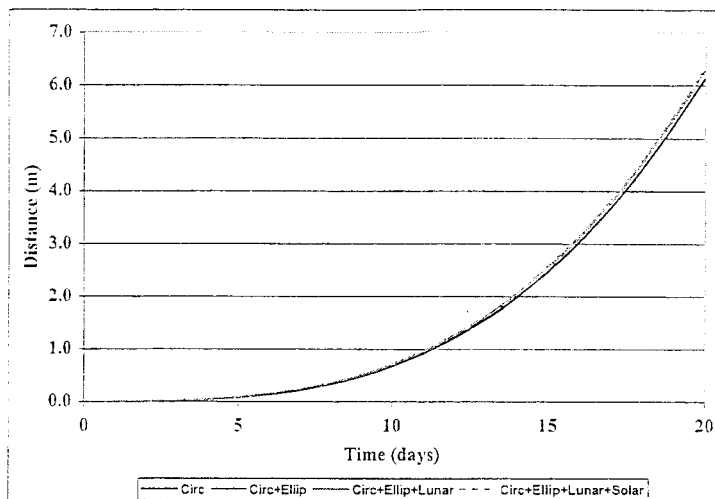


Figure 3
Effects of Perturbations on Relative Distance — Full Equations

EXPANDED EQUATIONS OF MOTION

Circular Restricted Three-Body Problem

In the previous research [1], Equation (1) was expanded through terms which are linear in the coordinates of r and no more than cubic in the coordinates of r_h .

The acceleration vector \ddot{r} may be written relative to a rotating coordinate system which rotates at the constant angular rate n about the z -axis normal to the ecliptic, and with the x direction as previously defined. This gives

$$\ddot{r} = \begin{bmatrix} \ddot{x} - 2n\dot{y} - n^2x \\ \ddot{y} + 2n\dot{x} - n^2y \\ \ddot{z} \end{bmatrix}$$

Elliptical Restricted Three-Body Problem

Consider the elliptical restricted three-body equations of motion as given in Equation (5). The right side of this set of relative acceleration equations may be expanded as in the earlier research. Consider the effects of various contributions to the magnitude ordering scheme. For ordering purposes, take

$$\begin{aligned}\rho &= 3.04 \times 10^{-6} \\ \gamma &= 1.01 \times 10^{-2} \\ \frac{r_h}{D} &= 4.0 \times 10^{-3} \quad (r_h = 600,000 \text{ km}) \\ \frac{r}{D} &= 3.2 \times 10^{-9} \quad (r = 0.5 \text{ km}).\end{aligned}$$

A rough estimate may be obtained of the contributions that the various perturbing contributions make to the circular restricted problem solution. Say that a perturbing term may be treated as modifying the linear frequencies associated with the circular restricted problem, and consider the square of the perturbing frequency to be roughly the magnitude of the coefficient of r in the perturbing acceleration. (The linear periods in and out of the xy -plane are approximately 177.566 days and 184.002 days, respectively.) Then, after 90 days, the effects of the terms containing various powers of r and r_h are given in Table 2, with effects included of roughly 20 m and larger.

Table 2
Along-Ellipsoid Effect of Sun-Earth Perturbation on Solution (90 days)

perturbing term	effect (m)
re	16.7
rr_h	289.3
rr_h^2	118.5
rr_h^3	47.4
rr_h^4	18.9

Retaining terms which contribute to approximately 20 m, again through linear terms in the relative motion and cubic terms in the hub position, results in the expanded/truncated differential equations

$$\begin{aligned}\ddot{\mathbf{r}} &= A_3 [-\mathbf{r} + 3x\hat{\mathbf{x}}] \\ &+ A_4 [3x\mathbf{r}_h + 3x_h\mathbf{r} + (3\mathbf{r}_h \cdot \mathbf{r} - 15xx_h)\hat{\mathbf{x}}] \\ &+ A_5 [(3\mathbf{r}_h \cdot \mathbf{r} - 15xx_h)\mathbf{r}_h + \frac{3}{2}(r_h^2 - 5x_h^2)\mathbf{r} - \frac{15}{2}(2x_h\mathbf{r}_h \cdot \mathbf{r} - 7xx_h^2 + xr_h^2)\hat{\mathbf{x}}] \\ &+ A_6 [\frac{15}{2}\mathbf{r}_h(-xr_h^2 - 2x_h\mathbf{r} \cdot \mathbf{r}_h + 7xx_h^2) + \frac{5}{2}\mathbf{r}(7x_h^3 - 3x_hr_h^2) \\ &\quad + \frac{15}{2}(-r_h^2\mathbf{r}_h \cdot \mathbf{r} + 7xr_h^2x_h + 7x_h^2\mathbf{r} \cdot \mathbf{r}_h - 21xx_h^3)\hat{\mathbf{x}}],\end{aligned}$$

where the constants A_i , tabulated in Table 5, are given by

$$A_i = \frac{\mu_1}{(x_e + D_1)^i} + \frac{\mu_2}{(x_e - D_2)^i}. \quad (9)$$

Next consider the left side of the differential equation:

$$\begin{aligned}\ddot{\mathbf{r}} &= \begin{bmatrix} \ddot{x} - \ddot{f}y - 2\dot{f}\dot{y} - \dot{f}^2x \\ \ddot{y} + \ddot{f}x + 2\dot{f}\dot{x} - \dot{f}^2y \\ \ddot{z} \end{bmatrix} \\ &= \begin{bmatrix} \ddot{x} - 2n_c\dot{y} - n_c^2x \\ \ddot{y} + 2n_c\dot{x} - n_c^2y \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} -\ddot{f}y - 2(\dot{f} - n_c)\dot{y} - (\dot{f}^2 - n_c^2)x \\ \ddot{f}x + 2(\dot{f} - n_c)\dot{x} - (\dot{f}^2 - n_c^2)y \\ 0 \end{bmatrix}. \quad (10)\end{aligned}$$

In this expanded form, the first vector term represents the acceleration which appears in the circular restricted problem (n_c refers to the mean motion of the circular restricted Earth orbit). The second vector term gives the perturbation which is added by including the elliptic restricted effects. The perturbation term is to be expanded in terms of the eccentricity e of Earth's orbit (≈ 0.017). In keeping with the earlier magnitude ordering, it is estimated that it is sufficient to retain only contributions which are at most linear in e , as er is approximately 8.5 m.

Say that the circular restricted problem takes D as being the reference value \bar{D} . Then, the corresponding mean motion is $n_c = \sqrt{\mu/\bar{D}^3}$. Now, in the elliptic restricted problem, the mean motion is

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{\mu}{\bar{D}^3} \frac{\bar{D}^3}{a^3}} = \left(\frac{\bar{D}}{a}\right)^{3/2} n_c.$$

If \bar{D} is chosen to be the semi-major axis a , then $n = n_c$. Alternatively, if \bar{D} is taken as the mean Sun-Earth distance $a(1 + e^2/2)$, the linear eccentricity approximation still allows n to be approximated by n_c .

Using two-body relationships, and linearizing in e , the time-varying \dot{f} is now approximated by

$$\dot{f} = \frac{h}{D^2} \approx n_c \left(\frac{a}{D}\right)^2 \approx n_c(1 + 2e \cos f)$$

or, in terms of the mean anomaly ℓ ,

$$\dot{f} \approx n_c(1 + 2e \cos \ell).$$

Differentiating,

$$\ddot{f} \approx -2en_c^2 \sin \ell.$$

Using these results in the perturbing vector of Equation (10) gives

$$\begin{bmatrix} -\ddot{f}y - 2(\dot{f} - n_c)\dot{y} - (\dot{f}^2 - n_c^2)x \\ \ddot{f}x + 2(\dot{f} - n_c)\dot{x} - (\dot{f}^2 - n_c^2)y \\ 0 \end{bmatrix} = \begin{bmatrix} -2eyn_c^2 \sin \ell - 4ejn_c \cos \ell - 4exn_c^2 \cos \ell \\ 2exn_c^2 \sin \ell + 4ejn_c \cos \ell - 4eyn_c^2 \cos \ell \\ 0 \end{bmatrix}.$$

Lunar Gravitational Effects

Consider the lunar force given above by Equation (7). It is desired that this contribution be expressed in terms of the position of the telescope relative to the hub. Let the vector \mathbf{r} again refer to the relative telescope position. Therefore,

$$\rho_{3t} = \rho_{3h} + \mathbf{r}. \quad (11)$$

As in the earlier analysis of the effects of Sun and Earth, the contribution of Equation (7) may be written as an expansion in terms of \mathbf{r} and its components. Accordingly, a similar binomial expansion development is followed. The square of the magnitude of ρ_{3t} is given by

$$\rho_{3t}^2 = \rho_{3t} \cdot \rho_{3t} = \rho_{3h}^2 \left[1 + \left(\frac{\mathbf{r}}{\rho_{3h}} \right)^2 + \frac{2\rho_{3h} \cdot \mathbf{r}}{\rho_{3h}^2} \right].$$

Using this formulation to perform a binomial expansion of Equation (7), using Equation (11) to substitute for ρ_{3t} , and expanding through linear terms in \mathbf{r} gives

$$\mathbf{F}_m \approx -\mu_3 \frac{\mathbf{r}}{\rho_{3h}^3} + 3\mu_3 \frac{\rho_{3h}}{\rho_{3h}^3} \frac{\mathbf{r} \cdot \rho_{3h}}{\rho_{3h}^2}. \quad (12)$$

It is preferable to treat these terms as an additive perturbation to the equations of the elliptical restricted three-body problem. Assume that the baseline elliptical restricted problem contains an object with combined terrestrial and lunar mass, located at the mean Earth-Moon barycenter. Therefore, the expansion of Equation (7) should contain the lunar contribution to this combined mass along with terms representing the effect of the lunar motion about the barycenter. Accordingly, Equation (12) may be rewritten as

$$\begin{aligned} \mathbf{F}_m \approx & -\mu_3 \left(\frac{\mathbf{r}}{\rho_{2h}^3} - \frac{3\rho_{2h} \mathbf{r} \cdot \rho_{2h}}{\rho_{2h}^3 \rho_{2h}^2} \right) \\ & - \mu_3 \mathbf{r} \left(\frac{1}{\rho_{3h}^3} - \frac{1}{\rho_{2h}^3} \right) + 3\mu_3 \mathbf{r} \cdot \left(\frac{\rho_{3h} \rho_{3h}}{\rho_{3h}^5} - \frac{\rho_{2h} \rho_{2h}}{\rho_{2h}^5} \right). \end{aligned} \quad (13)$$

Because the vector ρ_{2h} was used in the earlier analysis, it is convenient to write ρ_{3h} as

$$\rho_{3h} = \rho_{2h} + \mathbf{r}_{em},$$

where \mathbf{r}_{em} refers to the lunar position relative to the mean barycenter. Using a Legendre polynomial expansion, the inverse of the magnitude of ρ_{3h} is given by

$$\frac{1}{\rho_{3h}} = \frac{1}{\rho_{2h}} \sum_{k=0}^{\infty} \left(\frac{\tau_{em}}{\rho_{2h}} \right)^k P_k \left(\frac{\rho_{2h} \cdot \mathbf{r}_{em}}{\rho_{2h} \tau_{em}} \right).$$

Expanding through terms linear in τ_{em} , and substituting in Equation (13),

$$\begin{aligned} \mathbf{F}_m \approx & -\mu_3 \left(\frac{\mathbf{r}}{\rho_{2h}^3} - \frac{3\rho_{2h} \mathbf{r} \cdot \rho_{2h}}{\rho_{2h}^3 \rho_{2h}^2} \right) \\ & - 3\mu_3 \left[\mathbf{r} \rho_{2h} \cdot \mathbf{r}_{em} + \mathbf{r} \cdot \rho_{2h} \mathbf{r}_{em} + \mathbf{r} \cdot \mathbf{r}_{em} \rho_{2h} \right. \\ & \left. - 5\mathbf{r} \cdot \rho_{2h} \rho_{2h} \rho_{2h} \cdot \mathbf{r}_{em} / \rho_{2h}^2 \right] / \rho_{2h}^5. \end{aligned} \quad (14)$$

Examination of Equation (14) indicates that the form of the first term in \mathbf{F}_m is identical to the corresponding terrestrial term derived in the earlier work. The only difference is that, here, the mass coefficient is μ_3 rather than μ_2 . This term represents the effect of a lunar mass collocated with the terrestrial mass; the remainder of \mathbf{F}_m represents the effect of the lunar motion about Earth. The first term may be added to the earlier work simply by replacing μ_2 with $\mu_2 + \mu_3$ in the relative equations of motion for the elliptical restricted three-body problem.

For the second term of Equation (14), the Earth-hub position vector ρ_{2h} is expanded in a fashion similar to that of the earlier work. In the context of the elliptical restricted problem using a reference location of L_2 ,

$$\rho_{2h} = \gamma \bar{D} \hat{\mathbf{x}} + \mathbf{r}_h$$

where γ and \bar{D} are constants as previously defined. The square of the magnitude of ρ_{2h} is given by

$$\rho_{2h}^2 = \rho_{2h} \cdot \rho_{2h} = \gamma \bar{D}^2 \left[1 + \left(\frac{r_h}{\gamma \bar{D}} \right)^2 + \frac{2x_h}{\gamma \bar{D}} \right],$$

where x_h is the x -component of \mathbf{r}_h . Once again, inverse powers of ρ_{2h} are formed using binomial expansions. Expanding through linear terms in r_h and substituting, \mathbf{F}_m becomes

$$\begin{aligned} \mathbf{F}_m \approx & -\mu_3 \left(\frac{\mathbf{r}}{\rho_{2h}^3} - \frac{3\rho_{2h}\mathbf{r} \cdot \rho_{2h}}{\rho_{2h}^5} \right) \\ & - 3\mu_3 [(-2xx_{em} + yy_{em} + zz_{em})\hat{\mathbf{x}} + (yx_{em} + xy_{em})\hat{\mathbf{y}} \\ & \quad + (zx_{em} + xz_{em})\hat{\mathbf{z}}] / (\gamma \bar{D})^4 \\ & - \mu_3 [-15x_{em}x_h\mathbf{r} + 3\mathbf{r}_h \cdot \mathbf{r}_{em}\mathbf{r} - 15x_{em}x_h \\ & \quad + 3\mathbf{r} \cdot \mathbf{r}_h\mathbf{r}_{em} + 3(\mathbf{r} \cdot \mathbf{r}_{em})(-5x_h\hat{\mathbf{x}} + \mathbf{r}_h) + 105xx_{em}x_h\hat{\mathbf{x}} \\ & \quad - 15xx_{em}\mathbf{r}_h - 15x\mathbf{r}_h \cdot \mathbf{r}_{em}\hat{\mathbf{x}} - 15x_{em}\mathbf{r} \cdot \mathbf{r}_h\hat{\mathbf{x}}] / (\gamma \bar{D})^5. \end{aligned} \tag{15}$$

Again, a rough estimate of the contributions that the various perturbing contributions make to the solution is presented. As was done for the inclusion of the ellipticity, the perturbing terms are treated as modifying the linear frequencies — that is, the coefficient of r in the perturbing acceleration. After 90 days, the effects of the dominant lunar terms, containing various powers of the relevant variables, are given in Table 3.

Table 3
Along-Ellipsoid Effect of Lunar Perturbation on Solution (90 days)

perturbing term	effect (m)
$r r_{em}$	2.3
$r r_{em}^2$	0.6
$r r_h r_{em}$	0.9

Summary

The force equations derived above are now combined, repeating the models for solar radiation pressure and thrust. The resulting equation models the expanded elliptical restricted three-body problem, incorporating lunar, solar radiation, and thrust perturbations. The

expanded differential equations of relative motion for a telescope spacecraft are given by:

$$\begin{aligned}
\ddot{\mathbf{r}} = & A_3[-\mathbf{r} + 3x\hat{\mathbf{x}}] + A_4[3xr_h - 3x_h\mathbf{r} + (3\mathbf{r}_h \cdot \mathbf{r} - 15xx_h)\hat{\mathbf{x}}] \\
& + A_5[(3\mathbf{r}_h \cdot \mathbf{r} - 15xx_h)\mathbf{r}_h + \frac{3}{2}(r_h^2 - 5x_h^2)\mathbf{r} \\
& \quad - \frac{15}{2}(2x_h\mathbf{r}_h \cdot \mathbf{r} - 7xx_h^2 + xr_h^2)\hat{\mathbf{x}}] \\
& + A_6[\frac{15}{2}\mathbf{r}_h(-xr_h^2 - 2x_h\mathbf{r} \cdot \mathbf{r}_h + 7xx_h^2) + \frac{5}{2}\mathbf{r}(7x_h^3 - 3x_hr_h^2) \\
& \quad + \frac{15}{2}(-r_h^2\mathbf{r}_h \cdot \mathbf{r} + 7xr_h^2x_h + 7x_h^2\mathbf{r} \cdot \mathbf{r}_h - 21xx_h^3)\hat{\mathbf{x}}] \\
& - 3\mu_3[(-2xx_{em} + yy_{em} + zz_{em})\hat{\mathbf{x}} + (yx_{em} + xy_{em})\hat{\mathbf{y}} \\
& \quad + (zx_{em} + xz_{em})\hat{\mathbf{z}}]/(\gamma\bar{D})^4 \\
& - \mu_3[-15x_{em}x_h\mathbf{r} + 3\mathbf{r}_h \cdot \mathbf{r}_{em}\mathbf{r} - 15x_{em}x_h\hat{\mathbf{x}} \\
& \quad + 3\mathbf{r} \cdot \mathbf{r}_h\mathbf{r}_{em} + 3(\mathbf{r} \cdot \mathbf{r}_{em})(-5x_h\hat{\mathbf{x}} + \mathbf{r}_h) + 105xx_{em}x_h\hat{\mathbf{x}} \\
& \quad - 15xx_{em}\mathbf{r}_h - 15x\mathbf{r}_h \cdot \mathbf{r}_{em}\hat{\mathbf{x}} - 15x_{em}\mathbf{r} \cdot \mathbf{r}_h\hat{\mathbf{x}}]/(\gamma\bar{D})^5 \\
& + \frac{1.0198 \times 10^{17} C_R A \sigma}{m \|(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h + \mathbf{r}\|^3} [(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h + \mathbf{r}] \\
& - \frac{1.0198 \times 10^{17} C_R A \sigma}{m \|(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h\|^3} [(\gamma + 1)\bar{D}\hat{\mathbf{x}} + \mathbf{r}_h] \\
& + \begin{bmatrix} c(\psi') & -s(\psi') & 0 \\ s(\psi') & c(\psi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix} \frac{\mathbf{F}_{thrust}}{m},
\end{aligned}
\quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{ER3B} \\ \\ \text{lunar} \\ \text{SRP}_t \\ \text{SRP}_h \\ \text{thrust} \end{array}$$

where $\psi' = \psi + f$.

The following example permits comparison of the relative contribution of the terms to the telescope motion. The initial conditions are the same as those presented earlier in Table 1. As in Figure 3, Figure 4 presents the solution to four different force models selected from the relative motion summary equation above. Where hub position is required, it was obtained from separate integration of the full, unexpanded hub equations from the previous section, using the same force model as for the relative motion.

Once again, the reference solution is represented in the figure by the x -axis. This refers to the circular restricted model case. The other models depicted in Figure 4 represent the addition of ellipticity (ER3B), lunar, and solar radiation (SRP) effects over 20 days. Solar radiation is again treated as being applied to both the telescope and to either a real or phantom spacecraft at the hub, with the same physical characteristics as the telescope.

SENSITIVITY TO HUB POSITION ERROR

One area of concern to mission planners involves the limitations of accuracy in knowledge of the hub position. In particular, as the relative motion equations are dependent upon the hub position, it is important to understand how sensitive the relative motion solutions are to errors in the hub position. To address this issue, a detailed linear variational analysis was performed, clearly demonstrating that the errors in telescope position relative to the hub, based on knowledge of hub position to approximately 1.7 km, are sufficiently small that they may be ignored.

Several sample tests of this behavior were conducted. For one test case, the initial conditions of Table 1 were used here as a nominal set of initial conditions for the hub and

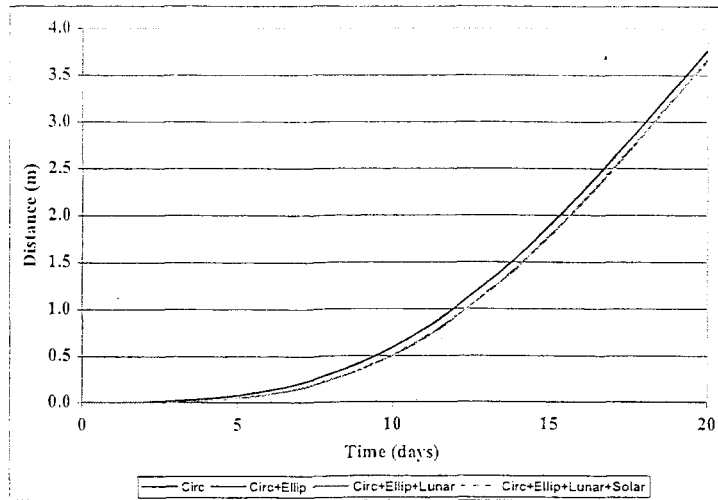


Figure 4
Effects of Perturbations on Relative Distance — Expanded Equations

telescope. The integration was then performed with the hub offset from its nominal initial state by 1 km in each directional component. This was to simulate an initial error in hub position. As seen in Figure 5, the telescope position relative to the hub differs from the nominal case by millimeters over 40 days.

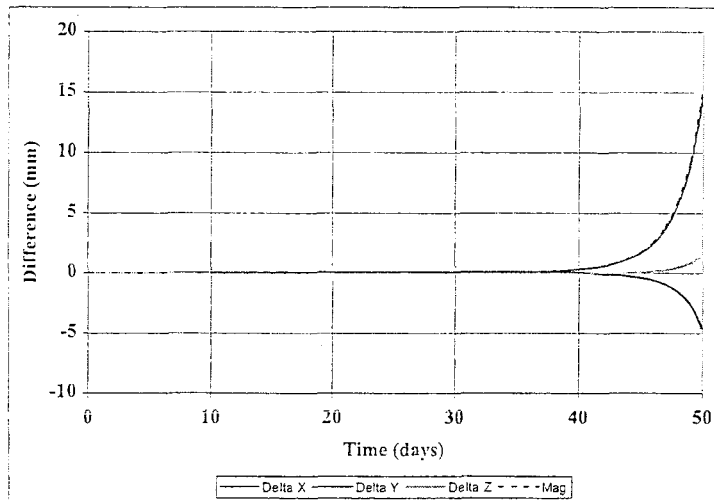


Figure 5
Relative Motion Error Caused by 1.7 km Initial Hub Position Error

The simulation was also repeated with the initial hub position offset from the nominal state by 17.3 km (10 km in each direction). The resulting relative motion error is roughly the same as that of Figure 5, scaled by a factor of 10.

SUMMARY AND CONCLUSION

This report details the further work describing the formation flying between spacecraft near the Sun-Earth L_2 libration point, beginning with the circular restricted three-body problem for the hub motion about L_2 .

These analyses develop the elliptical restricted three-body problem from previous work with circular problem [1]. The earlier in-depth work with the circular restricted problem was used as a basis from which to address the following perturbations:

- elliptical orbit of Earth-Moon about Sun
- lunar gravitational effects
- solar radiation pressure effects
- thrusters on vehicle

These were incorporated as additive perturbations to the circular restricted three-body problem with expansions of varying levels of fidelity. For use as a baseline, the equations of motion were first developed in their full nonlinear form; then, they were expanded through appropriately small contributions. The equations were implemented through their coding in a MATLAB simulation. One example was presented using the earlier identification of valid initial conditions that excite only oscillatory motion in the linear modes. The results shown in Figures 3 and 4 demonstrate the dominant perturbation due to the elliptical motion of Earth about the sun. In this example, the perturbations due to lunar and solar radiation pressure are negligible. Overall, the figure indicates that during the typical 5-day science observation of this example, these perturbations contribute less than 1 m of relative position difference. The proposed electric propulsion should have little trouble maintaining position.

The expanded circular and elliptical portions of the full model through terms which are linear in the coordinates of the telescope position relative to the hub and no more than cubic in the coordinates of the hub position relative to L_2 are presented above. The derivation describes the magnitude of the various terms and why some were truncated. Additionally, the lunar gravity model was expanded and some terms truncated due to very small perturbations upon the motion. As before, the perturbations due to the elliptical motion of the Earth about the Sun are dominant. Again, lunar and solar radiation pressure effects are negligible.

This work's uniqueness stems from its complete description of the primary perturbations to the relative motion between nearby spacecraft. In the course of the analysis, the dominance of the perturbing effect of the elliptical motion of the Earth about the sun was identified and verified. Contributions due to lunar gravity and solar radiation pressure are nearly negligible in the chosen examples.

CONSTANT PARAMETERS

Table 4
PHYSICAL CONSTANTS [3]
(E-M bary. = Earth-Moon barycenter)

gravitational parameter, Sun alone	μ_1	132,712,440,017.987 km ³ /s ²
gravitational parameter, Earth alone	μ_2	398,600.4415 km ³ /s ²
gravitational parameter, Earth-Moon	μ_2	403,503.236 km ³ /s ²
gravitational parameter, Moon alone	μ_3	4,902.8003 km ³ /s ²
astronomical unit	AU	149,597,870.691 km
mean E-M bary. distance from Sun		1.000001018 AU
eccentricity of E-M bary. orbit about Sun	e	0.01670862
mean motion of E-M bary. orbit about Sun	n	$0.199106385 \times 10^{-6}$ rad/s
L_2 distance ratio	γ	0.01007824

Table 5
COMPUTED VALUES AND COEFFICIENTS

reference Sun-Earth distance	\bar{D}	149,618,905.218739 km
gravitational coefficients	A_3	$1.16556055765939 \times 10^{-3}$ 1/day ²
(see Equation (9))	A_4	$5.84525170441422 \times 10^{-10}$ 1/(km-day ²)
	A_5	$3.86396147244215 \times 10^{-16}$ 1/(km ² -day ²)
	A_6	$2.56240418152728 \times 10^{-22}$ 1/(km ³ -day ²)

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